ON A CLASS OF TWO-INDEX REAL HERMITE POLYNOMIALS

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ABSTRACT. We introduce a class of doubly indexed real Hermite polynomials and we deal with their related properties like the associated recurrence formulae, Runge's addition formula, generating function and Nielsen's identity.

1. INTRODUCTION

The Burchnall's operational formula ([1])

$$\left(-\frac{d}{dx}+2x\right)^m (f) = m!\sum_{k=0}^{m}\frac{(-1)^k}{k!}\frac{H_{m-k}(x)}{(m-k)!}\frac{d^k}{dx^k}(f), \tag{1.1}$$

where $H_m(x)$ denotes the usual Hermite polynomial ([2, 6])

$$H_m(x) = (-1)^m e^{x^2}\frac{d^m}{dx^m}\left(e^{-x^2}\right), \tag{1.2}$$

enjoy a number of remarkable properties. It is used by Burchnall [1] to give a direct proof of Nielsen's identity ([4])

$$H_{m+n}(x) = m!n!\sum_{k=0}^{min(m,n)}\frac{(-2)^k}{k!}\frac{H_{m-k}(x)}{(m-n)!}\frac{H_{n-k}(x)}{(n-k)!}. \tag{1.3}$$

The special case of (1.1) where $f=1$, i.e.,

$$H_m(x) = \left(-\frac{d}{dx}+2x\right)^m \cdot (1). \tag{1.4}$$

can be employed to recover in a easier way the generating function

$$\sum_{m=0}^{+\infty} H_m(x)\frac{t^m}{m!} = \exp(2xt-t^2) \tag{1.5}$$

as well as the Runge addition formula ([5, 3])

$$H_m(x+y) = \left(\frac{1}{2}\right)^{m/2} m!\sum_{k=0}^{n}\frac{H_k(\sqrt{2}x)}{k!}\frac{H_{m-k}(\sqrt{2}y)}{(m-k)!}. \tag{1.6}$$

In this note, we have to consider the following class of doubly indexed real Hermite polynomials

$$H_{m,n}(x) = \left(-\frac{d}{dx}+2x\right)^m \cdot (x^n), \tag{1.7}$$

and we derive some of their useful properties. More essentially, we discuss the associated recurrence formulae, Runge's addition formula, generating function and Nielsen's identity.

2. DOUBLY INDEXED REAL HERMITE POLYNOMIALS $H_{m,n}(x)$

By taking $f(x)=x^n$ in (1.1), we obtain

$$H_{m,n}(x) := \left(-\frac{d}{dx}+2x\right)^m (x^n) \tag{2.1}$$

$$= m!n!\sum_{k=0}^{min(m,n)}\frac{(-1)^k}{k!}\frac{x^{n-k}}{(n-k)!}\frac{H_{m-k}(x)}{(m-k)!}. \tag{2.2}$$

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It follows that $H_{m,n}(x)$ is a polynomial of degree $m+n$, since
$$Q(x) := H_{m,n}(x) - x^n H_m(x)$$
is a polynomial of degree $deg(Q) \leq n+m-2$. For the unity of the formulations, we shall define trivially
$$H_{m,n}(x) = 0$$
whenever $m < 0$ or $n < 0$. We call them doubly indexed real Hermite polynomials. Note that $H_{m,0}(x) = H_m(x)$, $H_{0,n}(x) = x^n$ and
$$H_{m,n}(0) = \begin{cases} 0 & m < n \\ (-1)^n \frac{m!}{(m-n)!} H_{m-n}(0) & m \geq n \end{cases}. \tag{2.3}$$
A direct computation using (2.1) gives rise to
$$H_{1,n}(x) = -nx^{n-1} + 2x^{n+1}$$
for every integer $n \geq 1$. Note also that, since $H_1(x) = 2x$, it follows
$$H_{m+1}(x) = \left(-\frac{d}{dx} + 2x\right)^m (H_1(x)) = \left(-\frac{d}{dx} + 2x\right)^m (2x) = 2H_{m,1}(x). \tag{2.4}$$
The first few values of $H_{m,n}$ are given by

$H_{m,n}$	$n=1$	$n=2$	$n=3$
$m=1$	$-1+2x^2$	$-2x+2x^3$	$-3x^2+2x^4$
$m=2$	$-6x+4x^3$	$2-10x^2+4x^4$	$6x-14x^3+4x^5$
$m=3$	$6-24x^2+8x^4$	$24x-36x^3+8x^5$	$-6+54x^2-48x^4+8x^6$

From (2.2), one can deduce easily the symmetry formula
$$H_{m,n}(-x) = (-1)^{n+m} H_{m,n}(x), \tag{2.5}$$
so that the $H_{m,n}(x)$ is odd (rep. even) if and only if $n+m$ is odd (resp. even). Furthermore, let mention that the Rodrigues formula for $H_{m,n}(x)$ reads
$$H_{m,n}(x) = (-1)^m e^{x^2} \frac{d^m}{dx^m}\left(x^n e^{-x^2}\right). \tag{2.6}$$
Indeed, this is evidently proved using
$$\left(-\frac{d}{dx} + 2x\right)^m \cdot (f) = (-1)^m e^{x^2} \frac{d^m}{dx^m}\left(e^{-x^2} f\right). \tag{2.7}$$
Therefore, these polynomials constitute a subclass of the generalized Hermite polynomials
$$H_m^\gamma(x,\alpha,p) := (-1)^m x^{-\alpha} e^{px^\gamma} \frac{d^m}{dx^m}\left(x^\alpha e^{-px^\gamma}\right). \tag{2.8}$$
considered by Gould and Hopper in [7]. In fact, we have $H_{m,n}(x) = x^n H_m^2(x,n,1)$.

Proposition 2.1. *The polynomials $H_{m,n}$; $m,n \geq 1$, satisfy the following recurrence formulae*
$$H'_{m,n}(x) + H_{m+1,n}(x) - 2xH_{m,n}(x) = 0, \tag{2.9}$$
$$H_{m,n}(x) + nH_{m-1,n-1}(x) - 2H_{m-1,n+1}(x) = 0, \tag{2.10}$$
$$H_{m,n}(x) + mH_{m-1,n-1}(x) - xH_{m,n-1}(x) = 0, \tag{2.11}$$
$$(m-n)H_{m-1,n-1}(x) + 2H_{m-1,n+1}(x) + xH_{m,n-1}(x) = 0. \tag{2.12}$$

Proof. The first one follows by writing the derivation operator as
$$\frac{d}{dx} = -\left(-\frac{d}{dx} + 2x\right) + 2x.$$
Indeed, we get
$$\begin{aligned}\frac{d}{dx}\left(H_{m,n}(x)\right) &= -\left(-\frac{d}{dx} + 2x\right) H_{m,n}(x) + 2xH_{m,n}(x) \\ &= -H_{m+1,n}(x) + 2xH_{m,n}(x).\end{aligned}$$

For the second one, one writes $H_{m,n}(x)$ as

$$\begin{aligned} H_{m,n}(x) &= \left(-\frac{d}{dx}+2x\right)^{m-1}(H_{1,n}(x)) \\ &= \left(-\frac{d}{dx}+2x\right)^{m-1}\left(-nx^{n-1}+2x^{n+1}\right) \\ &= -nH_{m-1,n-1}(x)+2H_{m-1,n+1}(x). \end{aligned}$$

To prove (2.11), we use (2.6) combined with Leibnitz formula. Indeed,

$$\begin{aligned} H_{m,n}(x) &= (-1)^m e^{x^2}\frac{d^m}{dx^m}\left(x\cdot x^{n-1}e^{-x^2}\right) \\ &= (-1)^m e^{x^2}\left[x\frac{d^m}{dx^m}\left(x^{n-1}e^{-x^2}\right)+m\frac{d^{m-1}}{dx^{m-1}}\left(x^{n-1}e^{-x^2}\right)\right] \\ &= xH_{m,n-1}(x)-mH_{m-1,n-1}(x). \end{aligned}$$

Finally, (2.12) follows from (2.10) and (2.11) by substraction. □

Remark 2.2. *According to* (2.4)*, the* (2.11) *(corresponding to $n=1$) leads to the well known recurrence formula $H_{m+1}(x)=2xH_m(x)-2mH_{m-1}(x)$ for $H_m(x)$. Note also that* (2.9) *reduces further to $H'_m(x)+H_{m+1}(x)-2xH_m(x)=0$ by taking $n=0$.*

Proposition 2.3. *We have the following addition formula*

$$H_{m,n}(x+y)=m!n!\left(\frac{1}{\sqrt{2}}\right)^{m+n}\sum_{k=0}^{m}\sum_{j=0}^{n}\frac{H_{k,j}(\sqrt{2}x)}{k!j!}\frac{H_{m-k,n-j}(\sqrt{2}y)}{(m-k)!(n-j)!}. \tag{2.13}$$

Proof. We have

$$\begin{aligned} H_{m,n}\,(x+y) &= \left(-\frac{d}{d(x+y)}+2(x+y)\right)^m.((x+y)^n) \\ &= \left(-\frac{1}{2}\left(\frac{\partial}{\partial x}+\frac{\partial}{\partial y}\right)+2(x+y)\right)^m.((x+y)^n) \\ &= \left(\frac{1}{\sqrt{2}}\right)^m\left(A_x+A_y\right)^m.((x+y)^n) \\ &= \left(\frac{1}{\sqrt{2}}\right)^m\sum_{j=0}^{n}\binom{n}{j}\left(A_x+A_y\right)^m.(x^jy^{n-j}), \end{aligned}$$

where A_t stands for $A_t=-\partial/(\partial\sqrt{2}t)+2\sqrt{2}t$. Thus, since A_x and A_y commute, we can make use of the binomial formula to get

$$H_{m,n}\,(x+y)=\left(\frac{1}{\sqrt{2}}\right)^m\sum_{k=0}^{m}\sum_{j=0}^{n}\binom{m}{k}\binom{n}{j}A_x^k.(x^j)A_y^{m-k}.(y^{n-j}),$$

whence, we obtain the asserted result according to the fact that

$$A_t^r(t^s)=2^{-s/2}H_{r,s}(\sqrt{2}t).$$

□

Remark 2.4. *We recover the Runge addition formula* (1.6) *for the classical real Hermite polynomials $H_m(x)=H_{m,0}(x)$ by taking $n=0$ in* (2.13)*.*

The following identities are immediate consequence of the previous proposition.

Corollary 2.5. *The identity*

$$H_{m,n}(t)=m!n!\left(\frac{1}{\sqrt{2}}\right)^{m+n}\sum_{j=0}^{n}\sum_{k=j}^{m}\frac{(-1)^j}{j!(k-j)!}H_{k-j}(0)\frac{H_{m-k,n-j}(\sqrt{2}t)}{(m-k)!(n-j)!}$$

holds by taking $x=0$ and setting $t=y$ in (2.13)*, keeping in mind* (2.3)*. We get also*

$$H_{m,n}(t) = m!n!\left(\frac{1}{\sqrt{2}}\right)^{m+n}\sum_{k=0}^{m}\sum_{j=0}^{n}\frac{H_{k,j}(t/\sqrt{2})}{k!j!}\frac{H_{m-k,n-j}(t/\sqrt{2})}{(m-k)!(n-j)!}$$

by setting $x=y=t/2$ in (2.13)*. While for $t=-\sqrt{2}x=\sqrt{2}y$, we obtain*

$$\sum_{k=0}^{m}\sum_{j=0}^{n}(-1)^{k+j}\frac{H_{k,j}(t)}{k!j!}\frac{H_{m-k,n-j}(t)}{(m-k)!(n-j)!}=0$$

whenever $m+n$ is odd or $m>n$.

Next, we state the following

Proposition 2.6. *The generating function of $H_{m,n}$ is given by*

$$\sum_{m,n=0}^{+\infty} H_{m,n}(x)\frac{u^m}{m!}\frac{v^n}{n!} = \exp\left(-u^2+(2u+v)x-uv\right). \tag{2.14}$$

Proof. According to the definition of $H_{m,n}$, we can write

$$\begin{aligned}\sum_{m,n=0}^{+\infty} H_{m,n}(x)\frac{u^m}{m!}\frac{v^n}{n!} &= \left[\sum_{m=0}^{+\infty}\frac{1}{m!}\left(-u\frac{d}{dx}+2ux\right)^m\right]\cdot\left(\sum_{n=0}^{+\infty}\frac{v^n}{n!}x^n\right)\\ &= \exp\left(-u\frac{d}{dx}+2ux\right)(e^{vx}).\end{aligned}$$

Making use of the Weyl identity which reads for the operators $A=2xId$ et $B=-d/dx$ as

$$\exp(uA+uB)=\exp(uA)\exp(uB)\exp\left(-u^2Id\right); \quad u\in\mathbb{R},$$

we get

$$\sum_{m,n=0}^{+\infty} H_{m,n}(x)\frac{u^m}{m!}\frac{v^n}{n!} = e^{2ux-u^2}\exp\left(-u\frac{d}{dx}\right)(e^{vx}).$$

Therefore, the desired result follows since

$$\exp\left(-u\frac{d}{dx}\right)(e^{vx}) = \sum_{k=0}^{\infty}\frac{(-u)^k}{k!}\left(\frac{d}{dx}\right)^k(e^{vx}) = e^{-uv}e^{vx}.$$

□

Remark 2.7. *The special case of $v=0$ (in* (2.14)*) infers the generating function* (1.5) *of the standard real Hermite polynomials H_m. Furthermore, for $y=u=-v$, we get*

$$e^{xy} = \sum_{m,n=0}^{+\infty}(-1)^n H_{m,n}(x)\frac{y^{m+n}}{m!n!}. \tag{2.15}$$

Proposition 2.8. *We have the recurrence formula*

$$H'_{m,n}(x) = 2mH_{m-1,n}(x)+nH_{m,n-1}(x). \tag{2.16}$$

Proof. Differentiating the both sides of (2.14) and making appropriate changes of indices yield (2.16). □

Corollary 2.9. *We have*

$$\frac{d^\nu}{dx^\nu}(H_{r,n}(x)) = r!n!\sum_{j=0}^{\nu}\alpha_{j,\nu}\frac{H_{r-\nu+j,n-j}(x)}{(r-\nu+j)!(n-j)!}, \tag{2.17}$$

where

$$\alpha_{j,\nu} = \begin{cases} 2^\nu & \text{for } \; j=0\\ 2\alpha_{j,\nu-1}+\alpha_{j-1,\nu-1} & \text{for } \; 1\le j<\nu\\ 1 & \text{for } \; j=\nu\end{cases}.$$

Proof. This can be handled by mathematical induction using (2.16). □

Remark 2.10. *The $\alpha_{j,\nu}$ are even positive numbers and their first values are*

$\alpha_{j,\nu}$	$j=0$	$j=1$	$j=2$	$j=3$	$j=4$	$j=5$
$\nu=0$	1					
$\nu=1$	$\boxed{2}$	1				
$\nu=2$	2^2	$\boxed{4}$	1			
$\nu=3$	2^3	12	$\boxed{6}$	1		
$\nu=4$	2^4	32	24	$\boxed{8}$	1	
$\nu=5$	2^5	80	80	40	$\boxed{10}$	1

.

We conclude this note by giving a formula for the two-index Hermite polynomial $H_{m,n}(x)$ expressing it as a weighted sum of a product of the same polynomials. Namely, we state the following

Proposition 2.11. *Keep notation as above. Then the Nielsen identity for $H_{m,n}$; $n \geq 1$, reads*

$$H_{m+r,n}(x) = m!r!nn! \sum_{k,\nu,j=0}^{m,k,\nu} \alpha_{j,\nu} \frac{\Gamma(n+k-\nu)}{(k-\nu)!\nu!} \frac{(-x)^{\nu}}{x^{n+k}} \frac{H_{m-k,n}(x)}{(m-k)!n!} \frac{H_{r-\nu+j,n-j}(x)}{(r-\nu+j)!(n-j)!}.$$

Proof. Recall first that $H_m^{\gamma}(x,\alpha,p)$, the polynomials given through (2.8), can be rewritten in the following equivalent form ([7])

$$H_m^{\gamma}(x,\alpha,p) := \left(-\frac{d}{dx} + p\gamma x^{\gamma-1} - \frac{\alpha}{x}\right)^m (1).$$

Now, since for the special values $p=1$, $\gamma=2$ and $\alpha=n$, we have

$$\begin{aligned} H_{m+r,n}(x) &= x^n H^2_{m+r}(x,n,1) \\ &= x^n \left(-\frac{d}{dx} + 2x - \frac{n}{x}\right)^m \left(H_r^2(x,n,1)\right) \\ &= x^n \left(-\frac{d}{dx} + 2x - \frac{n}{x}\right)^m \left(x^{-n} H_{r,n}(x)\right), \end{aligned}$$

we can make use of the Burchnall's formula extension proved by Gould and Hopper [7], to wit

$$\left(-\frac{d}{dx} + p\gamma x^{\gamma-1} - \frac{\alpha}{x}\right)^m (f) = m! \sum_{k=0}^{m} \frac{(-1)^k}{k!} \frac{H^{\gamma}_{m-k}(x,\alpha,p)}{(m-k)!} \frac{d^k}{dx^k}(f).$$

Thus, for $f = x^{-n} H_{r,n}$, we obtain

$$H_{m+r,n}(x) = m! \sum_{k=0}^{m} \frac{(-1)^k}{k!} \frac{H_{m-k,n}(x)}{(m-k)!} \frac{d^k}{dx^k}(x^{-n} H_{r,n}(x)). \tag{2.18}$$

Therefore, by applying the Leibnitz formula and appealing the result of Corollary 2.9, we get

$$\begin{aligned} H_{m+r,n}(x) &= m! \sum_{k=0}^{m} \frac{(-1)^k}{k!} \frac{H_{m-k,n}(x)}{(m-k)!} \sum_{\nu=0}^{k} \binom{k}{\nu} \frac{d^{k-\nu}}{dx^{k-\nu}}(x^{-n}) \frac{d^{\nu}}{dx^{\nu}}(H_{r,n}(x)) \\ &= m!r!nn! \sum_{k,\nu,j=0}^{m,k,\nu} \alpha_{j,\nu} \frac{\Gamma(n+k-\nu)}{(k-\nu)!\nu!} \frac{(-x)^{\nu}}{x^{n+k}} \frac{H_{m-k,n}(x)}{(m-k)!n!} \frac{H_{r-\nu+j,n-j}(x)}{(r-\nu+j)!(n-j)!} \end{aligned}$$

for every integer $n \geq 1$. Note that for $n=0$, (2.18) reads simply

$$H_{m+r}(x) = m! \sum_{k=0}^{m} \frac{(-1)^k}{k!} \frac{H_{m-k}(x)}{(m-k)!} \frac{d^k}{dx^k}(H_r(x)).$$

In this case, we recover the usual Nielsen formula (1.3) for the real Hermite polynomials H_m. $\square$

References

[1] J. L. Burchnall, A note on the polynomials of Hermite. Quart. J. Math., Oxford Ser. 12, (1941). 9-11.
[2] C. Hermite, Sur un nouveau dveloppement en srie des fonctions. Compt. Rend. Acad. Sci. Paris 58, p. 94-100 et 266-273, t. LVIII (1864) ou Oeuvres compltes, tome 2. Paris, p. 293-308, 1908.
[3] J. Kamp de Friet, Sur une formule d'addition des polynomes d'Hermite. Volume 2 de Mathematisk-fysiske Meddelelser. 10 pages, Det Kgl. Danske Videnskabernes Selskab, Lunos, 1923
[4] N. Nielsen, Recherches sur les polynmes d'Hermite. Volume 1 de Mathematisk-fysiske meddelelser. 79 pages, Det Kgl. Danske Videnskabernes Selskab, 1918.
[5] C. Runge, ber eine besondere Art von Intergralgleichungen, Math. Ann. 75 (1914) 130-132.
[6] E.D. Rainville, Special functions. Chelsea Publishing Co., Bronx, N.Y., 1971.
[7] H.W. Gould, A.T. Hopper, Operational formulas connected with two generalizations of Hermite polynomials. Duke Math. J. 29 1962 51-63.

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